Research

A simple Statistical Mechanical Description of the Atmosphere Composed of Small Particles around Massive Spherical Body

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Abstract:
The atmosphere of small particles around a spherical massive body is theoretically described starting from statistical mechanical principles combined with the field approach. It is assumed that the system is in thermodynamic equilibrium and that the small particles are explicitly independent and therefore subjected to the Boltzmann statistics. The model system is the atmosphere of small particles that are attracted to the spherical massive body while their complete approach to the surface of this body is hindered by thermal motion. The gravitational attraction between the molecules and the massive object is described by introducing a potential of the attractive field and by considering that the massive object is the source of the field. The above assumptions lead to the formulation of the variational problem based on the 2.nd law of thermodynamics in the form of the consistently-related system of differential equations for the gravitational potential and the distribution of the molecules within the atmosphere.

Keywords: Gravitation; Atmosphere; Gravitational potential

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1 Introduction

The objective of modeling atmosphere in deeper space is the study of extraterrestrial bio-environment including molecules, micro and nano-organisms, and extracellular particles and vesicles. Formally similar models were constructed to model ionic atmosphere around charged bodies (see for example Hill, 1986). In this work we derive the expressions for the gravitational potential, density of the number of small particles and gravitational field in a system composed of a central spherical massive body surrounded by very many small particles which are subjected to thermal motion.

2 Theory

2.1 Field of the mass m

We take that the source of the field $\mathbf{G}$ is mass m,

$$\nabla \cdot \mathbf{G} = \varrho, \tag{1}$$

where

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V}, \tag{2}$$

and V is the volume. For convenience of scaling, we introduce a constant γ so that

$$\mathbf{G} = \Gamma\gamma \tag{3}$$

and

$$\nabla \cdot \mathbf{\Gamma} = \frac{\varrho}{\gamma}, \tag{4}$$

We assume that The field $\mathbf{\Gamma}$ has no vortexes,

$$\nabla \times \mathbf{\Gamma} = 0, \tag{5}$$

and introduce gravitational potential φ_Γ,

$$\mathbf{\Gamma} = \nabla\varphi_\Gamma. \tag{6}$$

Gradient in spherical coordinates reads

$$\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{1}{r\sin\varphi}\frac{\partial}{\partial \varphi}). \tag{7}$$

Combining Eqs.(4) and (6) yields the Poisson equation

$$\nabla^2\varphi_\Gamma = \frac{\rho}{\gamma}. \tag{8}$$

2.2 Poisson-Boltzmann equation

Assuming the thermodynamic equilibrium in a system with constant temperature and volume, and the validity of the Boltzmann statistics, the equality of gravitochemical potential at two different points (one at a chosen distance r from the massive body's center and the other at the surface of the massive body) reads

$$-m_1\varphi_\Gamma + kT\ln(\frac{\rho}{\rho_0}) = -m_1\varphi_\Gamma(r_0), \tag{9}$$

where ρ_0 is the density of small particles at the surface of the massive body, k is the Boltzmann constant and T is the temperature. After some rearranging of Eq.(9) we get the Boltzmann distribution

$$\rho = \rho_0 \exp\left(\frac{m_1}{kT}(\varphi_\Gamma - \varphi_\Gamma(r_0)\right). \tag{10}$$

The expression for ρ (Eq.(10)) is inserted into Eq.(8) to obtain the Poisson-Boltzmann differential equation for φ_Γ

$$\nabla^2\varphi_\Gamma = \frac{\rho_0}{\gamma} \exp\left(\frac{m_1}{kT}(\varphi_\Gamma - \varphi_\Gamma(r_0)\right). \tag{11}$$

As for the spherical geometry, we use the spherical coordinate system so that Eq.(11) transforms into

$$\frac{1}{r}\frac{\mathrm{d}^2(r\varphi_\Gamma)}{\mathrm{d}r^2} = \frac{\rho_0}{\gamma} \exp\left(\frac{m_1}{kT}(\varphi_\Gamma - \varphi_\Gamma(r_0)\right) \tag{12}$$

or

$$\frac{1}{r}\frac{\mathrm{d}^2(r(\varphi_\Gamma - \varphi_\Gamma(r_0)))}{\mathrm{d}r^2} = \frac{\rho_0}{\gamma} \exp\left(\frac{m_1}{kT}(\varphi_\Gamma - \varphi_\Gamma(r_0)\right) \tag{13}$$

2.3 Consistently related solution of the Poisson-Boltzmann equation

Solving the Poisson-Boltzmann equation (Eq.(13)) yields the gravitational potential φ_Γ in dependence on r, which is then used to calculate the distribution of small particles ρ and the gravitational field $\mathbf{\Gamma}$. For convenience, we introduce dimensionless quantities

$$y = \frac{m_1(\varphi_\Gamma - \varphi_\Gamma(r_0))}{kT}, \tag{14}$$

$$x = \frac{r}{r_0}, \tag{15}$$

and

$$\rho = \rho(r_0)\exp(y). \tag{16}$$

The Poisson-Boltzmann equation (11) transforms into

$$\frac{1}{x}\frac{\mathrm{d}^2(xy)}{\mathrm{d}x^2} = \kappa^2 \exp(y), \tag{17}$$

where

$$\kappa = \sqrt{\frac{\rho_0 r_0^2 m_1}{kT\gamma}}. \tag{18}$$

This is a nonlinear differential equation of the second order and to our best knowledge, it does not have an analytic solution. We explore the limiting case where y is small and the exponential function can be expanded,

$$\exp y = 1 + y, \tag{19}$$

so that

$$\frac{1}{x}\frac{\mathrm{d}^2(xy)}{\mathrm{d}x^2} = \kappa^2(1+y). \tag{20}$$

Eq.(20) can be solved analytically,

$$y = C_1\frac{\exp(-\kappa x)}{x} + C_2\frac{\exp(\kappa x)}{x} - 1, \tag{21}$$

where C_1 and C_2 are constants. As the density of small particles in not expected to rise to infinity far away from the massive sphere, C_2 is set to 0,

$$C_2 = 0, \tag{22}$$

so that

$$y = C_1 \frac{\exp(-\kappa x)}{x} - 1. \tag{23}$$

At the surface of the massive body, $x = 1$ and we chose the value of the potential to be 0. Therefore,

$$C_1 = \exp(\kappa), \tag{24}$$

so that

$$y = \frac{\exp(-\kappa(x-1))}{x} - 1. \tag{25}$$

It follows from Eq.(25) and Eq.(16) that

$$\rho = \rho_0 \exp(\frac{\exp(-\kappa(x-1))}{x} - 1). \tag{26}$$

The dimensionless gravitational field is obtained by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\kappa x - 1)}{x^2} \exp(-\kappa(x-1)). \tag{27}$$

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2.4 Estimation of parameters

In the integral form, Eq.(1) reads

$$\oint \mathbf{G} \cdot \mathrm{d}\mathbf{S} = m, \tag{28}$$

where $\mathbf{S}$ is the area. In the case of spherical massive body with mass m, the surface around the mass is taken to be a spherical shell. In this case, only the radial component of the vector $\mathbf{\Gamma} = (\Gamma_r, \Gamma_\theta, \Gamma_\phi)$ will give a nonzero contribution

$$\Gamma_r \gamma 4\pi r^2 = m, \tag{29}$$

so that

$$\Gamma_r = \frac{m}{4\pi\gamma r^2}, \tag{30}$$

where

$$\frac{1}{4\pi\gamma} = G \tag{31}$$

is the gravitation constant $G = 6.67 \times 10^{-11}$ $\mathrm{m}^3/(\mathrm{kg\ s}^2)$. Then,

$$\gamma = \frac{1}{4\pi G} = 1.2 \times 10^9 \frac{\mathrm{kg}^2}{\mathrm{Nm}^2}. \tag{32}$$

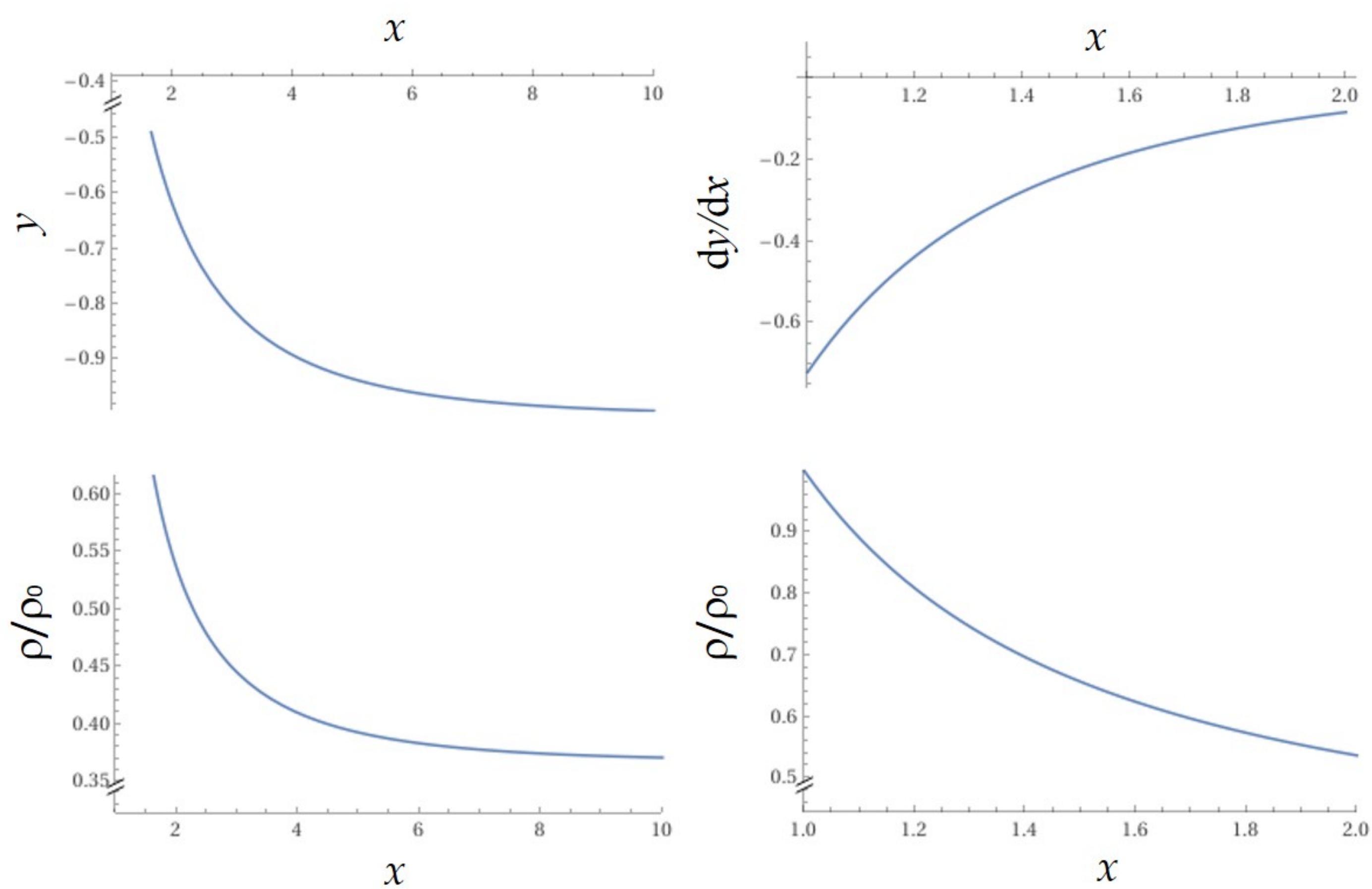

Figure 1: Calculated dimensionless gravitational potential y, dimensionless density of small particles ρ/ρ_0 and dimensionless gravitational field in the radial direction $\mathrm{d}y/\mathrm{d}x$ in dependence on the distance from the center of the spherical massive body x for $\kappa = 0.28$. The gravitational potential is the solution of the variational problem describing thermodynamic equilibrium of the atmosphere of small particles around the massive spherical body. The atmosphere develops from the surface of the massive body ($x = 1$) and expands to infinity.

3 Results

As an example we consider that the massive body has a mass of 6×10^{24} kg and radius is 6×10^{6}m. We take that the density of particles at the surface of the massive body is $\rho_0 = 1\mathrm{kg/m}^3$ and that the mass of the small particles m_1 is 10^{-26} kg. We take that the temperature is 280 K. Considering the above,

$$\kappa = \sqrt{\frac{\rho_0 r_0^2 m_1}{kT\gamma}} = 0.28. \tag{33}$$

Figure 1 shows calculated dimensionless gravitational potential y, dimensionless density of small particles ρ/ρ_0 and dimensionless gravitational field in the radial direction $\mathrm{d}y/\mathrm{d}x$ in dependence on the distance from the center of the spherical massive body x for $\kappa = 0.28$.

4 Conclusions

The field diminishes to 0 at large distances while the potential approaches a constant value as the value at the surface of the massive body was set according to the density of small particles at the surface. Consequently, in this model the density of the small particles diminishes, but does not vanish at large distances.

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6 Conflicts of Interest

The author discloses no conflict of interest.

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[1] Hill TL, An introduction to Statistical Thermodynamics. Dover Publications Inc. New York, 1986. PP: 321-331.