Research

Charging and Discharging a Capacitor: A Case Study in Solving Differential Equations with Separable Variables

Cafuta Kristijan[1,*]

[1.] University of Ljubljana, Faculty of Electrical Engineering, Laboratory of Applied Mathematics and Statistics, Ljubljana, Slovenia
* Correspondence: Kristijan Cafuta; kristijan.cafuta@fe.uni-lj.si

Citation: Cafuta K. Charging and Discharging a Capacitor: A Case Study in Solving Differential Equations with Separable Variables. Proceedings of Socratic Lectures. **2025**, 13, 130-137. https://doi.org/10.55295/PSL.13.2025.15

Abstract:
This article looks at the mathematical modelling of an electrical circuit that involves charging and discharging a capacitor. It is a great way to understand and solve first-order differential equations with separable variables. First, how to solve separable differential equations is explained. Second, a detailed derivation of the associated differential equations for discharging and charging are presented, followed by analytical solutions. This study shows how important mathematical ideas are used in solving real-world problems, especially in electrical engineering.

Keywords: Differential equations; Separable variables; Charging a Capacitor; Discharging a Capacitor

1. Introduction

Differential equations are very important tools for modelling lots of different physical problems. In electrical engineering, one example of this is the discharging of a capacitor through a resistor. This problem can be solved by using a first-order differential equation. Another similar example of this is charging a capacitor. This article uses these examples to show how differential equations with separable variables can be used to model and analyse dynamic systems.

2. Theoretical background: Separable differential equations

2.1. Definition

A first-order ordinary differential equation is called *separable* if it can be expressed in the form

$$y'(x) = f(x)g(y). \tag{1}$$

This means the derivative of y with respect to x can be written as a product of a function solely of x and a function solely of y.

2.2. Algorithm for solving separable differential equations

Separable differential equations can be solved in four steps.

Step 1. In the differential equation $y' = f(x)g(y)$ rewrite y' as $\frac{\mathrm{d}y}{\mathrm{d}x}$ and get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y). \tag{2}$$

Step 2. This allows us to separate the variables. Use algebra to change the equation to get all y-related terms with $\mathrm{d}y$ on one side and all x-related terms with $\mathrm{d}x$ on the other side of the equation:

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x. \tag{3}$$

Step 3. Integrating both sides of the equation independently yields:

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x. \tag{4}$$

Solve these integrals and, if possible, express y explicitly in terms of x.

Step 4. Use any given initial conditions to find the value of the constant of integration.

This technique is a simple way to solve many first-order ordinary differential equations. These equations are found in physical, biological and engineering applications. Separable equations are useful when modelling how a system changes over time gives a product of functions of the dependent and independent variables. As we said, this feature lets us separate the variables and then integrate both sides of the equation. Many pure mathematical examples can be found in (Boyce, 2017).

3. Example: Discharging a capacitor

Imagine a capacitor with capacitance C that is initially charged to a voltage U_0. It is then connected to a resistor with resistance R (see Figure 1). Our goal is to find the function $U_C(t)$ that describes how the voltage across this capacitor changes over time.

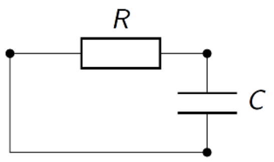

Figure 1. Discharging circuit diagram.

3.1. Required pre-knowledge

Tackling this problem requires some prior knowledge (Nilsson, 2023; Serway, 2019).

- Ohm's Law: The voltage across the resistor, U_R, equals the product RI (where I is the current)

$$U_R = RI \tag{5}$$

- Capacitor Relationship: The current I through a capacitor is related to the rate of change of voltage across it by

$$I = C\frac{\mathrm{d}U_C}{\mathrm{d}t} \tag{6}$$

- Second Kirchhoff's Law (Voltage Law): For a closed loop, the sum of voltages is zero. In our discharging circuit (with no external voltage source in the loop during discharge), this means

$$U_R + U_C = 0 \tag{7}$$

3.2. Modelling the RC circuit

We start with Kirchhoff's Law:

$$U_R + U_C = 0. \tag{8}$$

Substituting Ohm's Law yields:

$$RI + U_C = 0. \tag{9}$$

Substituting the capacitor relationship yields:

$$RC\frac{\mathrm{d}U_C}{\mathrm{d}t} + U_C = 0. \tag{10}$$

This is our (first-order linear homogeneous) differential equation describing the voltage across the capacitor during discharge.

3.3. Solving the discharging differential equation

First, we rearrange the equation to isolate the derivative term. This is a separable differential equation.

$$\frac{\mathrm{d}U_C}{\mathrm{d}t} = -\frac{1}{RC}U_C \tag{11}$$

We group terms involving U_C on one side and terms involving t on the other.

$$\frac{\mathrm{d}U_C}{U_C} = -\frac{1}{RC}\,\mathrm{d}t \tag{12}$$

Next, we integrate both sides. This yields:

$$\log|U_C| = -\frac{1}{RC}t + K, \tag{13}$$

where K is the constant of integration. Let us denote this constant as $\log(D)$ for convenience (and assuming $U_C > 0$):

$$\log(U_C) = -\frac{1}{RC}t + \log(D). \tag{14}$$

To solve this equation for U_C, we exponentiate both sides (we compute an antilogarithm).

$$U_C(t) = D\mathrm{e}^{-\frac{1}{RC}t} \tag{15}$$

Now, we apply the initial condition: at $t = 0$ s, the voltage is U_0. So, $U_C(0) = U_0$. This means $U_0 = D\mathrm{e}^0$ and therefore $D = U_0$.

Finally, the function describing the voltage across the discharging capacitor is:

$$U_C(t) = U_0\mathrm{e}^{-\frac{1}{RC}t}. \tag{16}$$

3.4. Graphs for discharging

The last obtained equation shows an exponential decay of voltage. The product RC is known as the “time constant” of the circuit, often denoted by τ. It determines how quickly the capacitor discharges. As you can see in Figure 2, if $RC = 7$ s, the voltage drops to a certain level. If RC is larger, say 10 s, the discharge is slower. Similarly, if you look at the second row in Figure 1, you can see that the initial voltage U_0 affects the whole curve. If U_0 is higher, the discharge starts from a higher voltage, but the shape of the decay, which is determined by RC, stays the same.

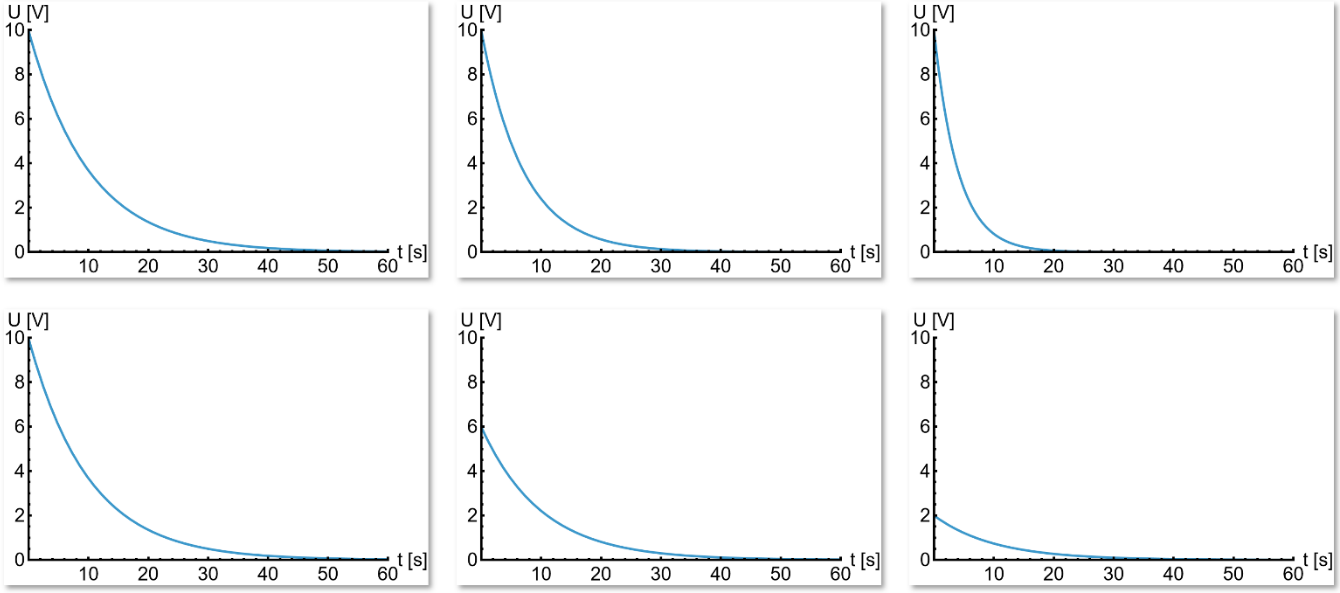

Figure 2. Discharging a capacitor. The first row shows how voltage decreases with different values of RC (10 s, 7 s, and 4 s with $U_0 = 10$ V) and the second row shows how voltage changes with different values of initial voltage U_0 (10 V, 6 V, and 2 V with $RC = 10$ s).

4. Example continues: Charging a capacitor

We revisit our RC circuit, but this time, we will look at charging a capacitor.

Imagine a capacitor with capacitance C, initially having zero voltage: $U_C(0) = 0$ V. At the time $t = 0$ s, we connect it to a resistor with resistance R and a DC voltage source with a constant voltage U_{source} (see Figure 3).

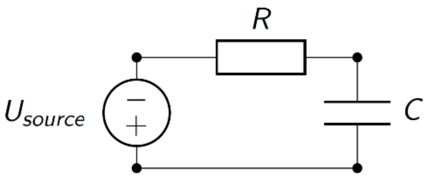

Figure 3. Charging circuit diagram.

We want to find the function $U_C(t)$ that describes how the voltage across the capacitor changes over time as it charges.

4.1. *What is different from the discharging case?*

- Ohm's Law $(U_R = RI)$ still holds.
- The capacitor relationship $\left(I = C\frac{\mathrm{d}U_C}{\mathrm{d}t}\right)$ still holds.
- However, Kirchhoff's Second Law now includes the voltage source:

$$U_R + U_C = U_{\text{source}}. \tag{17}$$

4.2. *Modelling the charging RC circuit*

We start with the Eq. (17) - modified Kirchhoff's Law.
Like in the discharging case, we use Ohm's law and substitute the capacitor relationship. We get

$$RC\frac{\mathrm{d}U_C}{\mathrm{d}t} + U_C = U_{\text{source}}. \tag{18}$$

4.3. *Solving the charging differential equation*

This is again a separable differential equation.
We can rearrange it as before (U_C on one side and t on the other side of the equation)

$$\frac{\mathrm{d}U_C}{U_\mathrm{source}-U_C}=\frac{1}{RC}\,\mathrm{d}t. \tag{19}$$

Integrating both sides yields

$$-\log|U_\mathrm{source}-U_C|=\frac{1}{RC}t-\log(D). \tag{20}$$

Next, we multiply both sides by -1, exponentiate both sides, and express U_C explicitly

$$U_C=U_\mathrm{source}-D\mathrm{e}^{-\frac{1}{RC}t}. \tag{21}$$

By applying the initial condition $U_C(0)=0$ V it follows $D=U_\mathrm{source}$ and therefore the final solution for charging a capacitor follows

$$U_C=U_\mathrm{source}\left(1-\mathrm{e}^{-\frac{1}{RC}t}\right). \tag{22}$$

4.4. *Graphs for charging*

The last obtained equation shows that the voltage across the capacitor increases from 0 and exponentially approaches the source voltage U_source (see Figure 4).

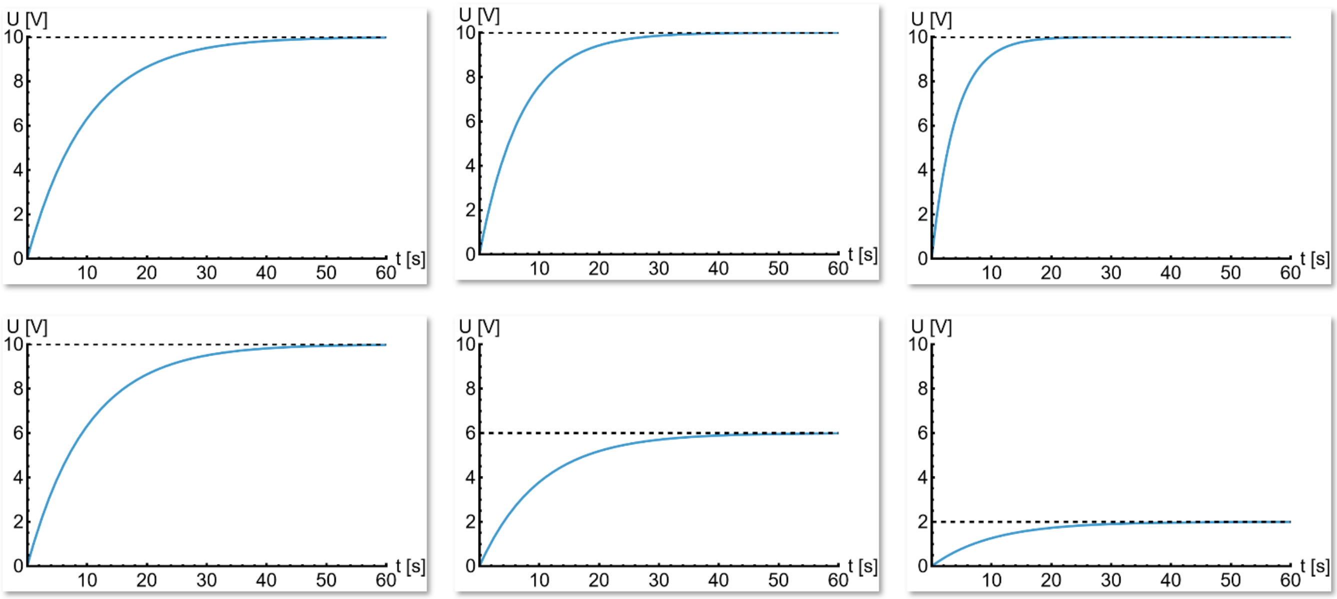

Figure 4. Charging a capacitor. The first row shows a rise of voltage with different values of RC (10 s, 7 s, and 4 s with $U_0=$ 10 V), and the second row shows different values of source voltage U_0 (10 V, 6 V, and 2 V with RC = 10 s). The dashed line shows the target voltage, U_source.

Again, the time constant RC determines how quickly this charging process happens. The dashed line shows the target voltage, U_source. You can see that for smaller RC values (e.g., $RC=4$ s), the capacitor charges faster towards U_source than for larger RC values (e.g.,

$RC = 10$ s). The different values of U_{source} simply set the maximum voltage the capacitor is trying to reach.

5. Some broader applications of separable differential equations

Separable differential equations are useful for much more than just RC circuits. You can use them to model many natural phenomena. Here are some examples. You can find even more examples in books and articles (e.g., Boyce, 2017; Braun, 1993; or Kreyszig, 2011).

5.1. Radioactive decay

The rate of decay of a radioactive substance $\frac{\mathrm{d}N(t)}{\mathrm{d}t}$ is proportional to the current amount of substance $N(t)$:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -k\,N(t). \tag{23}$$

The solution is

$$N(t) = N_0\,\mathrm{e}^{-kt}, \tag{24}$$

where N_0 is the initial amount and k is the decay constant.

5.1. Newton's law of cooling

The rate of change of temperature of an object $\frac{\mathrm{d}T(t)}{\mathrm{d}t}$ is proportional to the difference between its temperature $T(t)$ and the temperature of its surroundings $T_{\mathrm{surroundings}}$:

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -k\left(T(t) - T_{\mathrm{surroundings}}\right). \tag{25}$$

The solution is

$$T(t) = T_{\mathrm{surroundings}} + \left(T_0 - T_{\mathrm{surroundings}}\right)\mathrm{e}^{-kt}, \tag{26}$$

where T_0 is the initial temperature.

5.1. Population growth (unlimited resources)

In an environment with unlimited resources, the rate of population growth $\frac{\mathrm{d}N(t)}{\mathrm{d}t}$ can be proportional to the current population size $N(t)$:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = k\,N(t). \tag{27}$$

This leads to exponential growth:

$$N(t) = N_0\,\mathrm{e}^{kt}, \tag{28}$$

where N_0 is the initial population size.

5.1. Logistic growth (limited resources)

A more realistic population model considers a maximum carrying capacity N_{max}. The growth rate is then

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = k\,N(t)\left(N_{\mathrm{max}} - N(t)\right). \tag{29}$$

The solution is

$$N(t) = \frac{N_{\max}}{1 + \left(\frac{N_{\max} - N_0}{N_0}\right)\mathrm{e}^{-N_{\max} k t}} \tag{30}$$

where N_0 is the initial population size.
The solution shows that growth is slowing as it approaches the limit which is $N_{\max}$.

5. Conclusion

We have shown how we can define separable differential equations and how to solve them. We used this to model and analyse how a capacitor charges and discharges in a Resistor-Capacitor circuit. This way we derived the exponential decay and rise of voltage, respectively. Finally, we saw that these mathematical patterns are basic and appear in many different areas, from radioactive decay to population dynamics.
The key takeaway is the power of identifying a mathematical structure — like separable differential equations — and then use a method to find solutions that can be used to predict how things will behave in lots of different physical and biological systems.

Conflicts of interest: The author declares no conflict of interest.

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