

Research

Topology Enters Physics

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Abstract:

We illustrate Gauss-Bonnet and Poincare-Hopf (GB-PH) theorems in closed two-dimensional geometrical manifolds exhibiting in-plane orientational order which should be spatially homogeneous in equilibrium. Such geometrical shapes inevitable exhibit spatial dependent Gaussian curvature. The latter is a source of local frustrations that could stabilise topological defects (TDs) in the structure. Their presence is in general avoided because they introduce free energy penalties into the system. GB-PH theorem describes total winding number of TDs and indirectly their number in such systems. Namely, due to topological origin the behaviour of TDs obeys conservation laws which are determined by topological invariants. We present applicability of the GB-PH theorem in nematic liquid crystals, where such phenomena could be relatively easily tested experimentally. Furthermore, in terms of these theorems one could also describe quantum Hall effect, via which topology first entered physics.

Keywords: Topology; Gauss-Bonnet theorem; Poincare-Hopf theorem; Topological defects.

1. Introduction

Topology is a branch of mathematics, and it focuses on properties that are conserved under continuous transformations. Not long ago the Nobel prize winner Gamow pointed out that topology is one of rare examples whose topics are not yet exploited in explaining physical phenomena. Pioneering topological entrance in the realm of physics is the discovery of the integer Quantum Hall Effect (QHE) in 1980 (Tong, 2016). It was observed that Hall resistance R_H in two-dimensional (2D) electron systems are quantized - determined by an integer number n . A typical experimental set up is depicted in **Figure 1**.

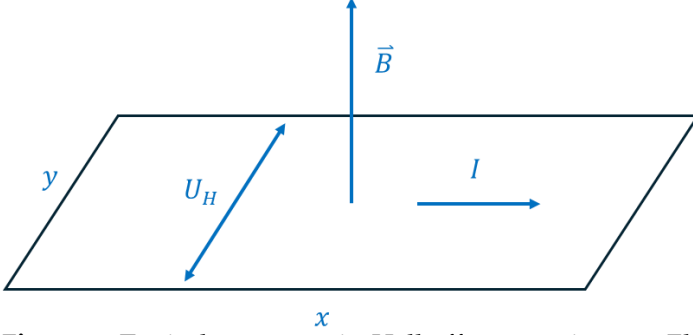
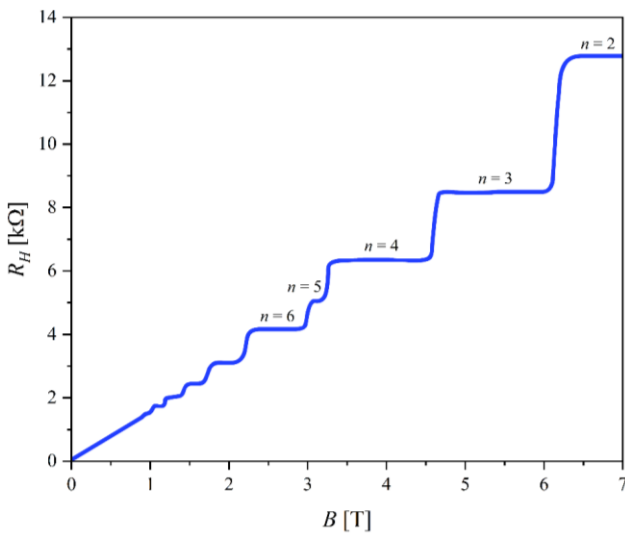


Figure 1. Typical geometry in Hall-effect experiments. Electron gas is confined to a 2D (xy) ribbon and the external magnetic field $\mathbf{B}$ is applied perpendicular to the ribbon larger surface. I denotes electric current and U_H denotes electric voltage.

The resulting QHE behaviour on increasing the magnetic field illustrated in **Figure 2**. The Hall resistance can be expressed as $R_H = h/(ne_0^2)$, where h is the Planck constant and e_0 is the elementary electric charge. Note that values of R_H are independent of material properties and consequently universal. The observed behaviour is robust, independent of microscopic imperfection because it is determined nonlocally, by the whole system. Such topological behaviours are of interest for various applications in physics. For example, it could be exploited for fault-tolerant quantum computing, where information is stored topologically (Han, 2023). This discovery opened the doors via which topology entered physics. From then on, it plays progressively increasing important role in physics. Clear evidence for this is the 2016 Nobel prize in physics (Gibney & Castelvechi, 2016) awarded to David J. Thouless, F. Duncan M. Haldane, and J. Michael Kosterlitz for their theoretical discoveries of topological phenomena in topological phases of matter. It might be that topology explains all observed discrete phenomena in nature (Wilczek, 2012).



In this contribution we present the Gauss-Bonnet (GB) and Poincare-Hopf (PH) topological theorem (do Carmo, 1976; Kamien, 2002) which play the central role in various physical phenomena. They could be derived using the universal parallel transport mechanism (Levi-Civita, 1916; Persico, 1921). For illustration let us consider 2D manifolds which exhibit in-plane order described by a vector field as depicted in **Figure 3**. The field is parallel transported along a curve whenever it is rigidly conveyed in a frame that displays the least motion capable of accompanying changes of the local unit normal. Let us consider closed paths in such manifolds and parallel transport the vector field along these paths. **Figure 3a** illustrates a loop on a flat substrate. One sees that in this case the initial orientation of the field and the final one, after completing the path $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$, are identical. However, if the enclosed surface exhibits a nonzero Gaussian curvature K_g (Kamien, 2002), this introduces a frustration in the system. Gaussian curvature for a regular surface patch is defined as $K_g = \kappa_1 \kappa_2$, where $\{\kappa_1 = 1/R_1, \kappa_2 = 1/R_2\}$ stand for the principal curvatures of the patch. Here R_1 and R_2 are the corresponding curvature radii (e.g., for a flat surface it holds $K_g = 0$, and $K_g = 1/R^2$ for a spherical surface of radius R). **Figure 3b** illustrates that in such cases the initial and final orientation are different because $K_g = 1/R^2$ of the enclosed ($A \rightarrow B \rightarrow C \rightarrow A$) surface patch is finite.

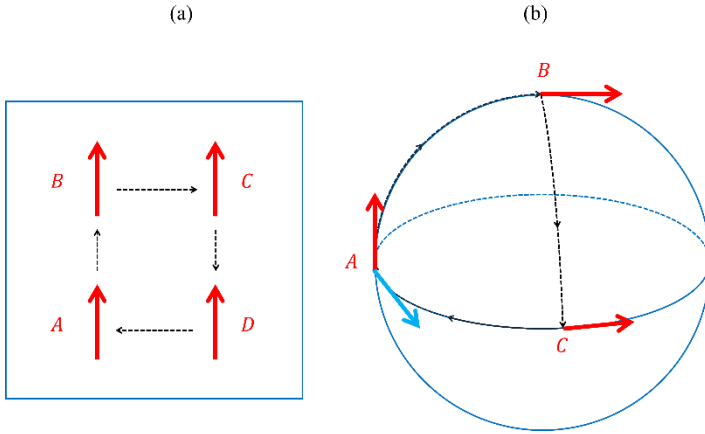


Figure 3. Parallel transport on 2D manifolds along the closed paths marked with dashed lines. a) Parallel transport $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ of a vector field on a flat manifold. The initial and final vector field orientation on traversing a closed path is identical. b) Parallel transport $A \rightarrow B \rightarrow C \rightarrow A$ on a sphere. The initial (red arrow) and final (blue arrow) orientation of the field are different because the enclosed surface exhibits nonzero Gaussian curvature.

In the following we present the GB-PH theorems on simple 2D geometrical manifolds exhibiting polar or axial orientation order. We chose cases where phenomena could be relatively easily visualised and understood. In conclusions we describe how these theorems could be generalised and their relationship with QHE.

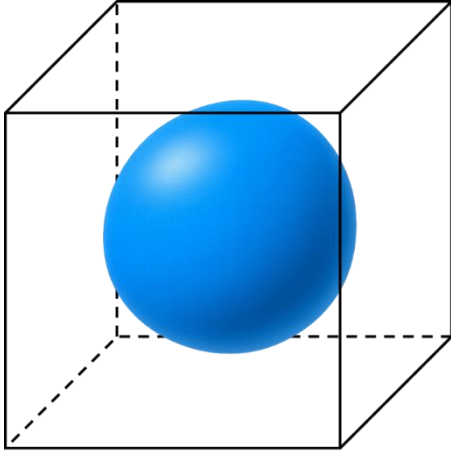
2. Results

In this section we present GB-PH theorems on 2D closed curved manifold possessing either vector or axial orientational order. We henceforth refer to such systems as *shells*. For the ease of understanding let us suppose that on flat substrates such order is spatially homogeneous in equilibrium. Such vector ordering field could represent thin closed ferromagnetic films. On the other hand, axial order is realised in thin nematic liquid crystal films (Rosso et al., 2012). A relatively simple derivation of GB-PH theorems for polar and axial fields in 2D manifolds is presented in Ref. 12.

The Gauss-Bonnet theorem (do Carmo, 1976) claims

$$\frac{1}{2\pi} \oint K_g dA = \chi_E = 2(1 - g). \quad (1)$$

Here, dA stands for an infinitesimally small surface area and the integration is carried out over the closed shell. The quantity χ_E is the Euler characteristic of the closed shape and g (genus) counts the number of holes within the system. For example, bodies exhibiting spherical or toroidal topology are characterised by $\{g=0, \chi_E = 2\}$ and $\{g=1, \chi_E = 0\}$, respectively. **Figure 4** illustrates how one could calculate χ_E of a shape by surrounding it with planar regular surfaces (Yang, 2012).



$$\chi_e = F - E + V$$

Figure 4. The Euler characteristic χ_E of a body. It is determined by the number F of surfaces ("faces"), number V of points ("vertices") and number E of lines ("edges") connecting the edges of a polyhedron within which the body is embedded. In the case shown $F=6$, $E=12$ and $V=8$ and $\chi_E = 2$.

Furthermore, the Poincare-Hopf theorem (do Carmo, 1976) claims

$$\chi_E = m_{\text{tot}}, \quad (2)$$

where m_{tot} stands for the total winding number of point topological defects (TDs) (Kurik & Lavrentovich, 1988) within the shell. TDs refer to the points where the order is not uniquely defined. There are characterised by the winding number m , which equals number of total rotations of the field on encircling the defect center once counterclockwise. Examples of TDs bearing $m = 1$ and $m = -1$ are shown in **Figure 5a** and **Figure 5b**. In 2D systems the winding number is equal to the 2D topological charge and is a conserved quantity. TDs bearing positive and negative values of m are commonly dubbed *defects* and *antidefects*, respectively. Namely, a pair $\{\text{defect}, \text{antidefect}\}$ could mutually annihilate into a defectless state, characterised by $m = 0$. **Figure 5c** illustrates a pair $\{\text{defect}, \text{antidefect}\}$ prior to annihilation. Namely, TDs are energetically expensive. Consequently, in general systems tend to be defectless, i.e., $m_{\text{tot}} = 0$. On a flat surface the pair shown in **Figure 5c** experiences an attractive force which would drive the system into a defectless, spatially homogeneous state. Note that due to energy reasons systems tend to form TDs bearing minimal winding numbers m_{min} (Kurik & Lavrentovich, 1988; Rosso et al., 2012). In case of polar and axial order it holds $|m_{\text{min}}| = 1$ and $|m_{\text{min}}| = 1/2$, respectively.

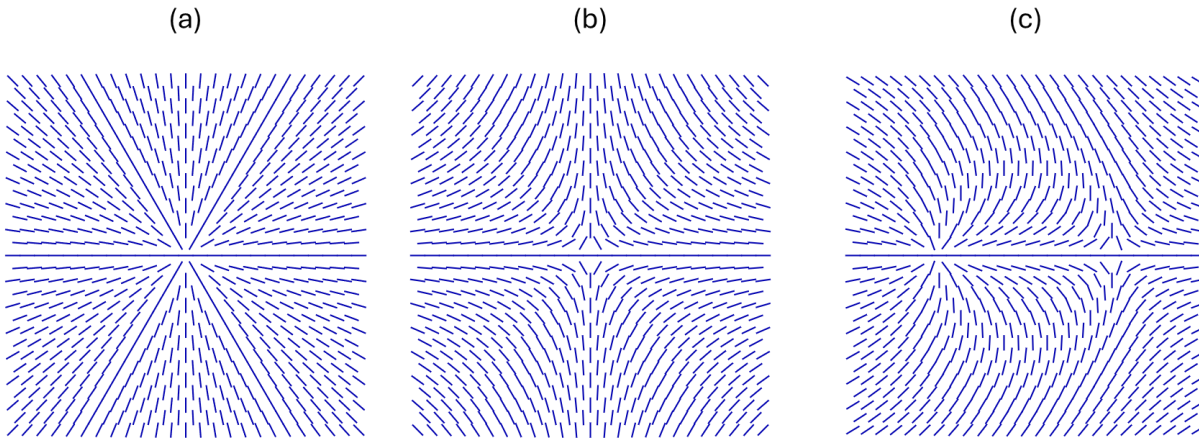


Figure 5. Topological point defects on a 2D flat surface. Single TDs characterised by (a) $m=1$ and (b) $m=-1$. (c) A pair $\{defect, antidefect\} = \{m=1, m=-1\}$ which is topologically equivalent to a defectless pattern.

However, closed shells exhibit in general surface patches that impose frustrations on ordering. This is clearly illustrated in **Figure 6**. In this case a spherical *shell* of radius R hosts orientational order. But due to the finite Gaussian curvature $K_g = 1/R^2$ the field could not be everywhere mutually parallel (such a case would be realised on a flat surface). Consequently, two $m=1$ TDs are formed at the poles of the structure, yielding $m_{tot} = 2$. Note that $K_g = 1/R^2$ yields $1/(2\pi) \oint K_g dA = 2$, fulfilling the GB-PH theorems (Eq.(1) and Eq.(2)).

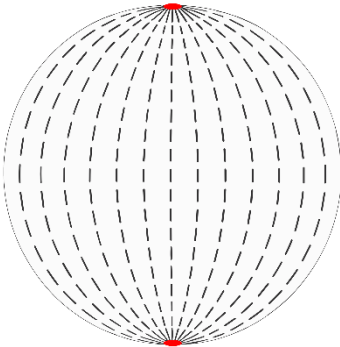


Figure 6. Topological point defects on a curved 2D spherical surface. The orientational pattern shown possesses two $m = 1$ point defects.

Experimentally, validity of these theorems could be relatively easily verified in nematic liquid crystals (NLCs) (Gharbi et al., 2013; Lopez-Leon et al., 2011). Namely, TDs in NLCs could be relatively easily created, manipulated and identified by polarising microscopy. Consequently, most of such studied are carried out in LCs. Some typical examples of resulting configurations are shown in **Figure 7**. These defect patterns were calculated numerically for a given *shell* geometry (Jesenek et al., 2015). One sees that in this case there are four $m=1/2$ on a spherical shell. Namely, the PH theorem requires $m_{tot} = 2$ and due to energy reason, the total winding number is distributed to 4 TDs bearing $m_{min} = 1/2$. Furthermore, in dumbbell-like shape one observes 8 TDs bearing $|m|=1/2$. There are two *antidefects* bearing $m = -1/2$, thus $m_{tot} = 2$ as requested for spherical topology. Note that in this case two additional pairs $\{defect, antidefect\} = \{1/2, -1/2\}$ are formed because of energy reasons. Namely, regions exhibiting $K_g > 0$ and $K_g < 0$ attract TDs bearing $m > 0$, and $m < 0$ (Jesenek et al., 2015) respectively.

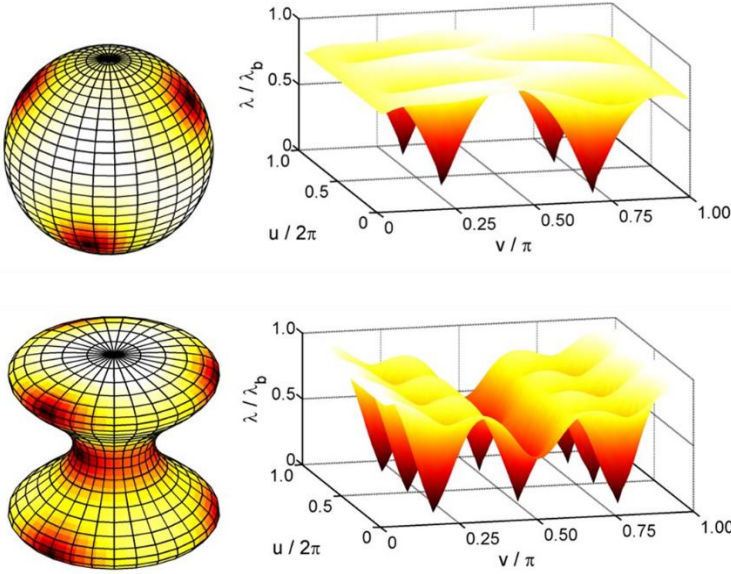


Figure 7. Topological point defects on a spherical (top panel) and dumbbell-like (bottom panel) shell. Left column: 3D view of surfaces hosting TDs. Right column: 2D view within the (u, v) plane, where u and v describe the azimuthal and zenithal angle describing the 2D shapes.

3. Conclusions

In the previous section we presented GB-PH theorems on geometric objects. We illustrated how these conservation laws are fingerprinted in topological invariants that remain unchanged under continuous transformations. However, similar behaviours could be observed also in other systems where 2D geometric shapes are described by some effective 2D physical field. For example, let us consider similar invariant physical properties for electrons in crystals. Their states are described by the Bloch wave function (Avron et al., 2003), $\psi = u_k e^{i\vec{k}\cdot\vec{r}}$. Here $u_k(\vec{r})$ exhibits the crystals translational symmetry and $\vec{k}$ describes the electron momentum $\vec{p} = \hbar\vec{k}$ with which electron hops from one atom to another and $\hbar = h/(2\pi)$. The associated de Broglie wavelength $\lambda = 2\pi/k$ cannot be smaller than the distance between neighbouring crystal cells. Consequently, the space of different momenta is restricted to the 1st Brillouin zone. Its shape reflects the arrangement of crystal atoms and plays the role of geometric space. Note that Brillouin zone exhibits periodic boundary conditions. To express topological theorem described in the previous section one needs to find an analogue of geometrical curvature. For this purpose one introduces Berry connection which is determined by u_k and related Berry curvature K_B , where details are given in (Avron et al., 2003). The resulting analogous topological conservation theorem reads

$$\frac{1}{2\pi} \oint K_B dA = n, \quad (3)$$

where $dA = d^2\vec{k}$ is the “surface” element of the 1st Brillouin zone and n is an integer, dubbed the Chern number. Furthermore, Avron et al. describe how similar approach is describing electrons in typical Hall geometrical setup show in **Figure 2** (Avron et al., 2003). This case illustrates how geometric topological theorems are present in a wide variety of physical phenomena where the relevant “shape geometry” could be described by some abstract physical field.

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