

Research

Derivation of Bending Energy of Thin, Mildly Curved, Laterally Isotropic Elastic Shell

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Citation: Kaljiglič V. Derivation of Bending Energy of Thin, Mildly Curved, Laterally Isotropic Elastic Shell. Proceedings of Socratic Lectures. 2026, 14, 102-107.
<https://doi.org/10.55295/PSL.14.2026.III10>

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Abstract:

Minimization of the energy of the system is a principle behind comprehension of stable membraneous structures, making the expression for the membrane energy a central problem in description of different systems. Here we present a derivation of the bending energy of a thin, mildly curved, laterally isotropic elastic shell from the basic assumptions regarding the deformation and stress. Using the equations of the differential geometry, the bending energy is expressed in terms of the invariants of the curvature tensor: trace and determinant. It is shown which assumptions are necessary to arrive to the result which is widely used in particular in membrane biophysics for description of the shapes of membrane-enclosed microscopic structures.

Keywords: Bending energy; Curvature; Differential geometry; Thin shells; Elastic modulus; Membrane

1 Free energy volume density of a deformed body

When impacted by external forces, the bodies deform. Deformation can be described in a way to give the change of the radius vector from the origin of the coordinate system to a chosen point on the body before deformation $\mathbf{r} = (x_1, x_2, x_3)$ and after the deformation $\mathbf{r}' = (x'_1, x'_2, x'_3)$,

$$\mathbf{u} = \mathbf{r}' - \mathbf{r}, \quad (1)$$

where $\mathbf{u}$ stands for the deformation vector. Written in components:

$$u_i = x'_i - x_i, \quad i = 1, 2, 3. \quad (2)$$

The distances between two nearby points before and after the deformation dl and dl' , respectively, are mutually connected,

$$dl'^2 - dl^2 + \sum_{i,k} 2\epsilon_{ik} dx_i dx_k, \quad (3)$$

where ϵ_{ik} stands for the deformation tensor $\underline{\epsilon}$. For small deformations,

$$\epsilon_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right). \quad (4)$$

It can be seen that $\underline{\epsilon}$ is a symmetrical tensor.

An external force $\mathbf{F}$ is acting on a chosen infinitesimal volume of an isotropic body,

$$\mathbf{F} = \int \mathbf{f} dV, \quad (5)$$

where $\mathbf{f}$ is the force density and dV is the volume element. Since the forces, which are causing deformation, are of a short range, the force $\mathbf{F}$ is acting on the chosen volume at the surface. Therefore integration over the volume can be replaced by the integration over the surface. Following the vector analysis, this can be done if we express the vector $\mathbf{f}$ as a divergence of a tensor of the 2.nd range,

$$f_i = \sum_{i,k} \frac{\partial \sigma_{ik}}{\partial x_k}, \quad (6)$$

where σ_{ik} is a component of the stress tensor. The stress tensor $\underline{\sigma}$ is symmetric,

$$\sigma_{ik} = \sigma_{ki}. \quad (7)$$

The change of the free energy volume density of the chosen volume of material due to bending is

$$dw_b = -s dT + \sum_{i,k} \sigma_{ik} d\epsilon_{ik}, \quad (8)$$

where s is the volume density of the entropy, and T is temperature. In the equilibrium at constant temperature, $dT = 0$,

$$\sigma_{ik} = \left(\frac{\partial w_b}{\partial \epsilon_{ik}} \right)_T. \quad (9)$$

We express the free energy volume density w_b in terms of the deformation tensor. We consider small deformations so that we can expand w_b in terms of elements of the deformation tensor. If

the membrane is not deformed at a constant temperature $\epsilon_{ik} = 0$, there is no stress $\sigma_{ik} = 0$. It follows from Eq.(9), that the expansion should not include linear terms $\epsilon_{i,k}$. We consider that like w_b , the expansion terms should be scalars. From the deformation tensor we can compose two independent scalars of the second order: $(\sum_i \epsilon_{ii})^2$ and $\sum_{i,k} \epsilon_{ik}^2$. Volume density of the free energy is

$$w_b = \frac{\lambda}{2} \left(\sum_i \epsilon_{ii} \right)^2 + \mu \sum_{i,k} \epsilon_{ik}^2, \quad (10)$$

where λ and μ are the Lamée coefficients. We express the deformation as a sum of squeezing and shear deformation,

$$\epsilon_{ik} = \left(\epsilon_{ik} - \frac{1}{3} \delta_{ik} \sum_l \epsilon_{ll} \right) + \frac{1}{3} \delta_{ik} \sum_l \epsilon_{ll}, \quad (11)$$

where $\delta_{ik} = 1$ when $i = k$ and 0 when $i \neq k$. Inserting Eq.(11) into Eq.(10) yields

$$w_b = \frac{K}{2} \left(\sum_i \epsilon_{ii} \right)^2 + \mu \sum_{i,k} \left(\epsilon_{ik} - \frac{1}{3} \delta_{ik} \sum_l \epsilon_{ll} \right)^2 \quad (12)$$

where $1/K$ is called compressibility

$$K = \lambda + \frac{2}{3} \mu. \quad (13)$$

The coefficient μ is called the shear modulus. In order for f_b to reach a minimum at $\epsilon_{ik} = 0$, the coefficients K and μ should both be positive. The coefficients are connected with the Young modulus E and Poisson coefficient ν (Landau and Lifshitz, 1986, Kralj-Iglič, 1993),

$$E = \frac{8K\mu}{(3K + \mu)}, \quad (14)$$

$$\nu = \frac{(3K - 2\mu)}{2(3K + \mu)}. \quad (15)$$

Since K and μ are always positive, also E is positive, while ν can have different values between

$$-1 \leq \nu \leq \frac{1}{2}. \quad (16)$$

2 Free energy volume density of a thin, mildly curved shell

We consider that the shell is very thin and that the displacement of the membrane from the plane xy around a chosen point on the shell is described by a function $\zeta(x, y)$,

$$u_z = \zeta(x, y). \quad (17)$$

We neglect external forces comparing to the stresses, so the stresses are only tangential,

$$\sigma_{xz} = \sigma_{zx} = 2\mu\epsilon_{zx} = 0, \quad (18)$$

$$\sigma_{xz} = \sigma_{zx} = 2\mu\epsilon_{zx} = 0, \quad (19)$$

$$\sigma_{zz} = -\frac{\lambda}{2} \left(\epsilon_{xx} + \epsilon_{yy} + \left(1 + \frac{2\mu}{\lambda}\right) \epsilon_{zz} \right) = 0. \quad (20)$$

If we insert Eq.(4) into Eqs.(18)-(20), we get

$$\frac{\partial u_x}{\partial z} = -\frac{\partial \zeta(x, y)}{\partial x}, \quad (21)$$

$$\frac{\partial u_y}{\partial z} = -\frac{\partial \zeta(x, y)}{\partial y}, \quad (22)$$

$$\epsilon_{zz} = -\frac{1}{(1 + 2\mu/\lambda)}(\epsilon_{xx} + \epsilon_{yy}). \quad (23)$$

We integrate Eqs.(24) and (25) from 0 to z and consider that at $z = 0$ it should be fulfilled

$$u_x = z \frac{\zeta(x, y)}{\partial x}, \quad (24)$$

$$u_y = -\frac{\partial \zeta(x, y)}{\partial y}. \quad (25)$$

With derivation of expressions (24) and (25) we yield

$$\epsilon_{xx} = z \frac{\partial^2 \zeta(x, y)}{\partial x^2}, \quad (26)$$

$$\epsilon_{yy} = z \frac{\partial^2 \zeta(x, y)}{\partial y^2}, \quad (27)$$

$$\epsilon_{xy} = z \frac{\partial^2 \zeta(x, y)}{\partial x \partial y}. \quad (28)$$

It follows from Eqs.(18) and (19) that

$$\epsilon_{xz} = \epsilon_{yz} = 0, \quad (29)$$

and from Eq.(23) that

$$\epsilon_{zz} = -\frac{\lambda z}{(\lambda + 2\mu)} \left(\frac{\partial^2 \zeta(x, y)}{\partial x^2} + \frac{\partial^2 \zeta(x, y)}{\partial y^2} \right). \quad (30)$$

The free energy volume density is obtained by inserting Eqs.(26) - (30) into Eq.(10). After some rearrangement, we get

$$f_b = z^2 \left(\frac{2\mu(\lambda + \mu)}{(\lambda + 2\mu)} \left(\frac{\partial^2 \zeta(x, y)}{\partial x^2} + \frac{\partial^2 \zeta(x, y)}{\partial y^2} \right)^2 + 2\mu \left(\left(\frac{\partial^2 \zeta(x, y)}{\partial x \partial y} \right)^2 - \frac{\partial^2 \zeta(x, y)}{\partial x^2} \frac{\partial^2 \zeta(x, y)}{\partial y^2} \right) \right). \quad (31)$$

Bending energy of the shell W_b is

$$W_b = \int \int \int f_b \, dx \, dy \, dz. \quad (32)$$

Integration is performed over the surface and the thickness of the shell.

3 Curvature of the surface

The surface is described by the radius vector $\mathbf{R} = (x, y, \zeta(x, y))$. The equations of differential geometry yield the expressions for the trace and the determinant of the diagonalized curvature tensor $\underline{C}$, the trace being $\text{Tr } \underline{C} = C_1 + C_2$ and $\text{Det } \underline{C} = C_1 C_2$,

$$C_1 + C_2 = \frac{(EN + GL - 2FM)}{(EG - F^2)}, \quad (33)$$

$$C_1 C_2 = \frac{(LN - M^2)}{(EG - F^2)}, \quad (34)$$

where C_1 and C_2 are the principal curvatures.

$$E = 1 + \left(\frac{\partial \zeta(x, y)}{\partial x} \right)^2, \quad (35)$$

$$F = \frac{\partial \zeta(x, y)}{\partial x} \frac{\partial \zeta(x, y)}{\partial y}, \quad (36)$$

$$G = 1 + \left(\frac{\partial \zeta(x, y)}{\partial y} \right)^2, \quad (37)$$

$$L = \frac{\frac{\partial^2 \zeta(x, y)}{\partial x^2}}{\sqrt{\left(1 + \frac{\partial^2 \zeta(x, y)}{\partial x^2} + \frac{\partial^2 \zeta(x, y)}{\partial y^2} \right)}}, \quad (38)$$

$$M = \frac{\frac{\partial \zeta(x, y)}{\partial x} \frac{\partial \zeta(x, y)}{\partial y}}{\sqrt{\left(1 + \frac{\partial^2 \zeta(x, y)}{\partial x^2} + \frac{\partial^2 \zeta(x, y)}{\partial y^2} \right)}}, \quad (39)$$

$$N = \frac{\frac{\partial^2 \zeta(x, y)}{\partial y^2}}{\sqrt{\left(1 + \frac{\partial^2 \zeta(x, y)}{\partial x^2} + \frac{\partial^2 \zeta(x, y)}{\partial y^2} \right)}}. \quad (40)$$

For small $\partial \zeta(x, y)/\partial x$,

$$EG - F^2 \simeq 1, \quad (41)$$

$$C_1 + C_2 \simeq \frac{\partial^2 \zeta(x, y)}{\partial x^2} + \frac{\partial^2 \zeta(x, y)}{\partial y^2} \quad (42)$$

and

$$C_1 C_2 = \frac{\partial^2 \zeta(x, y)}{\partial x^2} \frac{\partial^2 \zeta(x, y)}{\partial y^2} - \left(\frac{\partial^2 \zeta(x, y)}{\partial x \partial y} \right)^2. \quad (43)$$

The surface area element is

$$dA = \sqrt{(EG - F^2)}, \quad (44)$$

and the volume element is

$$dV = dx \, dy \, dz. \quad (45)$$

4 Bending energy of thin mildly curved membrane

Comparing the expression (31) with expressions (42) and (43), we recognize (Struik, 1961; Do Carmo, 2016),

$$W_b = \frac{1}{2}k_c \iint (C_1 + C_2)^2 dA + \frac{1}{2}k_G \iint C_1 C_2 dA, \quad (46)$$

where the elastic constants k_c and k_G are

$$k_c = \frac{2\mu(\lambda + \mu)h^3}{6(\lambda + 2\mu)}, \quad (47)$$

$$k_G = -\frac{1}{3}\mu h^3, \quad (48)$$

and h is the thickness of the membrane. If we express the elastic constants with the Young modulus E and the Poisson coefficient σ , we get

$$k_c = \frac{Eh^3}{12(1 - \sigma^2)}, \quad (49)$$

$$k_G = -\frac{Eh^3}{6(1 + \sigma)}. \quad (50)$$

Since the Young modulus is always positive and the Poisson coefficient cannot be smaller than -1 and larger than 1/2, it follows from Eqs.(16), (49), and (50) that the bending constant k_c is positive and the Gaussian constant k_G is negative.

5 Conclusions

Starting from deformation of the material due to forces acting on a surface, and taking into account the equations of differential geometry, we derived the bending energy of a thin, mildly curved elastic shell.

Funding: This work was funded by the Slovenian Research and Innovation Agency, core funding No. P3-0388, and project No. J3-60063.

Conflict of Interest: The author declares no conflict of interest.

References

- [1] Do Carmo MP. Differential Geometry of Curves and Surfaces (2nd Edition). Revised & Updated. 2016, pp: 157-160.
- [2] Kralj-Iglič V. Thesis. University of Ljubljana. 1993, pp: 63-71.
- [3] Landau LD, Lifshitz EM. Theory of Elasticity (Vol. 7 of Course of Theoretical Physics), 1986, pp: 38-42.
- [4] Struik DJ. Lectures on Classical Differential Geometry (2nd Edition). Dover Publications. 1961, pp: 117-120.