

Research

Equivalent Characterisations of Singularity in Curvature-Coupled Membrane Functionals

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Abstract:

We study a curvature-based functional describing the coupling between isotropic bending and curvature anisotropy in membrane shapes. The functional is reformulated in invariant geometric form, enabling a structural analysis from three complementary viewpoints: the quadratic curvature kernel, the linearised shape operator, and the Euler–Lagrange equations. In each formulation, a common threshold is identified at which the highest-order curvature contribution degenerates. This provides an equivalent characterisation of the resulting singular behaviour, linking it to a loss of ellipticity of the associated operator and to a vanishing of curvature stiffness in the deviatoric mode. The coincidence of these conditions across all three perspectives demonstrates that the singularity is intrinsic to the model and independent of the chosen representation. These results clarify the structural origin of degeneracy in curvature-coupled membrane theories.

Keywords: Curvature-based functionals; Membrane bending; Curvature anisotropy; Loss of ellipticity; Singular behaviour; Variational formulation

1 Introduction

The equilibrium shapes of fluid membranes are commonly described within the framework of curvature elasticity, where the bending energy is expressed in terms of the principal curvatures of the surface. In the classical Helfrich model (Helfrich, 1973; Deuling and Helfrich, 1976), the energy density depends quadratically on the mean curvature, leading to a variational problem governed by a fourth-order geometric operator. This approach successfully captures a wide range of membrane morphologies observed in lipid vesicles and related systems (Helfrich, 1973, Deuling and Helfrich, 1976, Seifert, 1997).

In many situations of physical and biological interest, however, the membrane exhibits anisotropic behaviour. This may arise from orientational ordering of lipid molecules, the presence of embedded proteins, or other structural inhomogeneities. Such effects introduce a coupling not only to the mean curvature but also to the curvature anisotropy, measured by the difference of the principal curvatures. This motivates the inclusion of additional terms in the energy functional that depend on the curvature deviator.

A specific model incorporating such effects was proposed in (Kralj-Iglič et al., 2006), based on a microscopic description of quadrupolar ordering of phospholipid molecules. In this formulation, the curvature energy density is given by

$$f = \frac{1}{4}(c_1 + c_2)^2 - \kappa \ln \cosh(2\vartheta d), \quad (1)$$

where c_1 and c_2 denote the principal curvatures of the surface normalized by the curvature of the sphere with the same area A , $c_1 = C_1 R_s$, $c_2 = C_2 R_s$, $A = 4\pi R_s^2$. The normalized mean curvature h and the normalized curvature deviator d are defined by

$$h = \frac{c_1 + c_2}{2}, \quad d = \frac{c_1 - c_2}{2}. \quad (2)$$

The parameter κ characterises the strength of the anisotropic interaction, while ϑ controls the sensitivity of this interaction to curvature anisotropy. The first term corresponds to the classical isotropic bending energy, whereas the second term introduces a nonlinear dependence on the curvature deviator.

This model predicts a rich variety of equilibrium shapes and, in particular, exhibits the formation of narrow neck regions in axisymmetric configurations. Numerical continuation results reported in (Kralj-Iglič et al., 2006) reveal the presence of parameter intervals in which the governing equations become singular and solutions cannot be continued. While this behaviour is clearly visible at the level of the axisymmetric shape equations, its structural origin is not immediately apparent.

In particular, it is not clear whether the observed singularity is a consequence of the chosen parametrisation, a limitation of the numerical method, or an intrinsic feature of the underlying variational formulation. Addressing this question requires an analysis that does not rely on a specific coordinate representation, but instead examines the structure of the geometric functional itself.

The aim of the present work is to provide such a structural interpretation. Rather than focusing on individual solutions, we analyse the curvature functional and identify the mechanism responsible for the loss of regularity. Our approach is based on three complementary viewpoints: the quadratic curvature kernel, which captures the highest-order contribution to the energy; the linearised shape operator, whose principal symbol determines ellipticity; and the full Euler–Lagrange equations, which describe the nonlinear equilibrium conditions.

We show that all three perspectives lead to the same threshold condition. At this threshold, the contribution associated with curvature anisotropy cancels the isotropic bending stiffness

in the deviatoric mode. As a result, the quadratic form loses coercivity, the principal symbol becomes non-positive, and the highest-order term in the Euler–Lagrange equations vanishes. This identifies the singular behaviour observed in (Kralj-Iglić et al., 2006) as a manifestation of a loss of ellipticity of the underlying geometric operator.

The present analysis thus provides a unified interpretation of the transition observed in the model, connecting its energetic, differential, and nonlinear aspects. In this way, the neck singularity can be understood as a structural consequence of the loss of curvature stiffness in the anisotropic mode.

2 Invariant form of the functional

Let C_1, C_2 denote the principal curvatures,

$$H = \frac{C_1 + C_2}{2}, \quad D = \frac{C_1 - C_2}{2}. \quad (3)$$

The functional derived in (Kralj-Iglić et al., 2006) can be written as

$$F(S) = \int (w_i(2H)^2 - w_a\Phi(D)) \, dA, \quad (4)$$

where

$$\Phi(D) = \ln \cosh(\alpha D), \quad (5)$$

w_i, w_a and α are constants, and S is the arclength of the contour.

The first (isotropic) term in Eq.(4) penalises mean curvature, while the second (deviatoric) term favours curvature anisotropy.

3 Connection to the dimensionless formulation

Using the dimensionless quantities,

$$f = w_b + f_d, \quad w_b = \frac{1}{4}(c_1 + c_2)^2, \quad f_d = -\kappa \ln \cosh(2\vartheta d), \quad (6)$$

with

$$h = \frac{c_1 + c_2}{2}, \quad d = \frac{c_1 - c_2}{2}. \quad (7)$$

The correspondence is

$$w_i = \frac{1}{4}, \quad w_a = \kappa, \quad \alpha = 2\vartheta. \quad (8)$$

4 Quadratic curvature kernel

To analyse the structural well-posedness of the functional, we isolate the highest-order quadratic dependence on curvature. This corresponds to extracting the leading contribution of the energy density in terms of the principal curvatures.

4.1 Expansion of the deviatoric term

The deviatoric contribution is given by

$$\Phi(D) = \ln \cosh(\alpha D). \quad (9)$$

For small D , we expand:

$$\ln \cosh(\alpha D) = \frac{\alpha^2}{2} D^2 + O(D^4). \quad (10)$$

Substituting into (4), the functional becomes, to quadratic order,

$$Q = \int \left(w_i (2H)^2 - w_a \frac{\alpha^2}{2} D^2 \right) dA. \quad (11)$$

This expression captures the highest-order curvature dependence and therefore determines the principal part of the associated Euler–Lagrange operator.

4.2 Representation in principal curvatures

We now rewrite the quadratic kernel in terms of the principal curvatures C_1, C_2 ,

$$H = \frac{C_1 + C_2}{2}, \quad D = \frac{C_1 - C_2}{2}. \quad (12)$$

Then

$$(2H)^2 = (C_1 + C_2)^2, \quad D^2 = \frac{(C_1 - C_2)^2}{4}. \quad (13)$$

Substituting into (11), we obtain

$$Q = \int \left[w_i (C_1 + C_2)^2 - \frac{w_a \alpha^2}{8} (C_1 - C_2)^2 \right] dA. \quad (14)$$

Expanding both terms,

$$(C_1 + C_2)^2 = C_1^2 + 2C_1 C_2 + C_2^2, \quad (15)$$

$$(C_1 - C_2)^2 = C_1^2 - 2C_1 C_2 + C_2^2. \quad (16)$$

Thus

$$Q = \int \left[\left(w_i - \frac{w_a \alpha^2}{8} \right) (C_1^2 + C_2^2) + \left(2w_i + \frac{w_a \alpha^2}{4} \right) C_1 C_2 \right] dA. \quad (17)$$

4.3 Mode decomposition

It is more instructive to diagonalise the quadratic form in curvature modes. We consider two fundamental modes: Mean curvature mode ($C_1 = C_2$) and deviatoric mode ($C_1 = -C_2$).

4.3.1 Mean curvature mode ($C_1 = C_2 = C$)

$$H = C, \quad D = 0, \quad (18)$$

$$Q \sim 4w_i C^2. \quad (19)$$

Thus the corresponding curvature stiffness, meaning the coefficient multiplying the quadratic curvature contribution in this mode, is denoted by λ_+ :

$$\lambda_+ = w_i. \quad (20)$$

4.3.2 Deviatoric mode ($C_1 = -C_2 = C$)

$$H = 0, \quad D = C, \quad (21)$$

$$Q \sim -\frac{w_a \alpha^2}{2} C^2. \quad (22)$$

Combining both contributions yields eigenvalues

$$\lambda_{\pm} = w_i \pm \frac{w_a \alpha^2}{4}. \quad (23)$$

Here λ_+ and λ_- are the eigenvalues of the quadratic curvature form after diagonalisation into the mean and deviatoric curvature modes. They represent the effective curvature stiffnesses associated with these two independent modes.

4.4 Coercivity condition

The quadratic form is positive definite if and only if both eigenvalues, namely the mean-mode eigenvalue λ_+ and the deviatoric-mode eigenvalue λ_- , are positive:

$$\lambda_+ > 0, \quad \lambda_- > 0. \quad (24)$$

This gives

$$-4w_i < w_a \alpha^2 < 4w_i. \quad (25)$$

Within this interval, the quadratic curvature kernel is coercive, and the highest-order contribution to the energy is strictly positive.

4.5 Interpretation of the loss of deviatoric curvature stiffness

The isotropic term always contributes positive stiffness through the mean curvature mode. In contrast, the deviatoric term acts as a *negative stiffness* in the anisotropic mode. As the parameter $w_a \alpha^2$ increases, it progressively cancels the isotropic bending stiffness in the deviatoric direction. At the threshold

$$w_a \alpha^2 = 4w_i, \quad (26)$$

the deviatoric stiffness vanishes:

$$\lambda_- = 0. \quad (27)$$

Beyond this point, the quadratic form becomes indefinite, and the curvature functional is no longer coercive in the deviatoric mode.

4.6 Connection to the dimensionless formulation

We now repeat the analysis in the dimensionless variables used in (Kralj-Iglič et al., 2006). The density reads

$$f = \frac{1}{4}(c_1 + c_2)^2 - \kappa \ln \cosh(2\vartheta d). \quad (28)$$

For small d ,

$$\ln \cosh(2\vartheta d) = 2\vartheta^2 d^2 + O(d^4). \quad (29)$$

Thus

$$f_{\text{quad}} = \frac{1}{4}(c_1 + c_2)^2 - 2\kappa\vartheta^2 d^2. \quad (30)$$

In terms of

$$h = \frac{c_1 + c_2}{2}, \quad d = \frac{c_1 - c_2}{2}, \quad (31)$$

this becomes

$$f_{\text{quad}} = 2h^2 - 2\kappa\vartheta^2 d^2. \quad (32)$$

The eigenvalues are therefore

$$\lambda_h = 1, \quad \lambda_d = 1 - 2\kappa\vartheta^2. \quad (33)$$

Positivity requires

$$2\kappa\vartheta^2 < 1. \quad (34)$$

4.7 Conclusion (kernel viewpoint)

Both invariant and dimensionless formulations lead to the same structural condition. The quadratic curvature kernel loses positivity when the deviatoric stiffness vanishes. This threshold will reappear in the analysis of the linearised operator and the Euler–Lagrange equations.

5 Coercivity and ellipticity

Positivity of the quadratic curvature kernel implies positivity of the principal part of the linearised operator. In particular, the fourth-order geometric operator is elliptic when both eigenvalues are positive (Evans, 2010; Gazzola, 2010). Thus ellipticity is equivalent to

$$\lambda_+ > 0, \quad \lambda_- > 0. \quad (35)$$

This establishes the structural threshold for well-posedness at the level of the quadratic kernel.

6 Interpretation of neck singularity

In the original axisymmetric formulation, singular behaviour appears in the curvature equations when a coefficient vanishes. From the present viewpoint, this corresponds to the loss of stiffness in the deviatoric mode, i.e. when

$$\lambda_- \rightarrow 0. \quad (36)$$

At this point, the quadratic kernel ceases to be positive definite and the operator loses ellipticity. Thus the neck singularity reflects a structural degeneration of the variational operator.

7 Linearised operator and principal symbol

We now analyse the same structural condition at the level of the linearised shape operator. While the quadratic curvature kernel determines the energetic coercivity, the linearised operator governs the differential structure of the Euler–Lagrange equations. In particular, ellipticity is determined by the principal symbol of this operator.

7.1 Normal variation of the surface

Let Σ be a smooth reference surface with unit normal n . Consider a normal variation

$$x \mapsto x + \eta n, \quad (37)$$

where η is a scalar perturbation field. To leading order, the variation of the mean curvature is given by

$$\delta H = -\Delta_{\Sigma} \eta + \text{lower-order terms}, \quad (38)$$

where Δ_{Σ} is the Laplace–Beltrami operator on the surface. The deviatoric curvature

$$D = \frac{C_1 - C_2}{2} \quad (39)$$

also depends linearly on second derivatives of η . To leading order, one may write

$$\delta D = \mathcal{D}_{\Sigma} \eta + \text{lower-order terms}, \quad (40)$$

where $\mathcal{D}_{\Sigma}$ is a second-order tangential operator. Its precise form depends on the local geometry, but its principal part is of the same differential order as Δ_{Σ} .

7.2 Second variation of the functional

We now substitute these variations into the quadratic part of the energy. From Section 4, the quadratic density is

$$f_{\text{quad}} = 2H^2 - 2\kappa\vartheta^2 D^2. \quad (41)$$

The second variation therefore contains terms of the form

$$\delta^2 F \sim \int_{\Sigma} [2(\delta H)^2 - 2\kappa\vartheta^2 (\delta D)^2] dA. \quad (42)$$

Substituting (38)–(40), we obtain, at leading order,

$$\delta^2 F \sim \int_{\Sigma} [2(\Delta_{\Sigma} \eta)^2 - 2\kappa\vartheta^2 (\mathcal{D}_{\Sigma} \eta)^2] dA. \quad (43)$$

Thus both contributions involve second derivatives of η , and the resulting operator is of fourth order [6].

7.3 Principal symbol

To determine ellipticity, we analyse the operator in local coordinates or Fourier space. Let k denote the tangential wavevector. Then:

$$\Delta_{\Sigma}\eta \longrightarrow -|k|^2\hat{\eta}, \quad (44)$$

$$\mathcal{D}_{\Sigma}\eta \longrightarrow \sigma_D(k)\hat{\eta}, \quad (45)$$

where $\sigma_D(k)$ is homogeneous of degree two in k . Consequently,

$$(\Delta_{\Sigma}\eta)^2 \sim |k|^4|\hat{\eta}|^2, \quad (46)$$

$$(\mathcal{D}_{\Sigma}\eta)^2 \sim \gamma(\theta)|k|^4|\hat{\eta}|^2, \quad (47)$$

where $\gamma(\theta)$ captures the angular dependence of the deviatoric mode in tangent space. The principal symbol of the linearised operator is therefore

$$\sigma(k) = (2 - 2\kappa\vartheta^2\gamma(\theta)) |k|^4. \quad (48)$$

7.4 Ellipticity condition

Ellipticity requires that the principal symbol be strictly positive for all nonzero k (Evans, 2010):

$$\sigma(k) > 0 \quad \text{for all } k \neq 0. \quad (49)$$

Since $\gamma(\theta)$ is bounded and achieves its maximum in the pure deviatoric direction, the worst-case coefficient is

$$2(1 - 2\kappa\vartheta^2). \quad (50)$$

Thus ellipticity holds if and only if

$$2\kappa\vartheta^2 < 1. \quad (51)$$

7.5 Interpretation of the loss of ellipticity

The operator consists of two competing fourth-order contributions:

- the isotropic bending term, proportional to $(\Delta_{\Sigma}\eta)^2$, which provides positive stiffness,
- the deviatoric term, proportional to $(\mathcal{D}_{\Sigma}\eta)^2$, which acts as a negative stiffness in the anisotropic mode.

As the parameter $\kappa\vartheta^2$ increases, the deviatoric contribution progressively cancels the isotropic bending stiffness. At the critical threshold

$$2\kappa\vartheta^2 = 1, \quad (52)$$

the principal symbol vanishes in the deviatoric direction. At this point, the operator is no longer strictly elliptic.

7.6 Conclusion (operator viewpoint)

The linearised operator loses ellipticity precisely at the same threshold identified in the quadratic curvature kernel. This confirms that the loss of coercivity at the energy level corresponds to a loss of ellipticity at the level of the governing differential operator.

8 Euler–Lagrange structure and degeneracy

We now analyse the same structural condition at the level of the full Euler–Lagrange equations. While the quadratic kernel determines energetic coercivity and the linearised operator determines ellipticity, the Euler–Lagrange system describes the fully nonlinear governing equations of equilibrium shapes.

8.1 Structure of the axisymmetric equations

In the original formulation of [4], the surface is represented by an axisymmetric meridian curve. The Euler–Lagrange equations obtained from the functional involve curvature-dependent coefficients multiplying the highest-order derivatives. In particular, the shape equations contain a factor of the form

$$B = 1 - \frac{2\kappa\vartheta^2}{\cosh^2(2\vartheta d)}, \quad (53)$$

where d is the deviatoric curvature. This factor multiplies the highest-order curvature terms in the differential equation governing the meridian profile. This degeneracy is consistent with the behaviour observed in earlier numerical studies of the same class of functionals [4]. In particular, a finite parameter interval exists in which the Euler–Lagrange system ceases to admit regular numerical solutions. Within this interval, the associated Lagrange multipliers tend to zero, indicating a loss of constraint enforcement at the level of the variational formulation, while the total energy remains smooth and does not exhibit any discontinuity. This separation between the regularity of the solution and the behaviour of the energy reflects the vanishing of the highest-order curvature contribution, rather than any energetic instability.

8.2 Degeneracy condition

The function $\cosh^2(x)$ satisfies

$$\cosh^2(x) \geq 1 \quad \text{for all } x. \quad (54)$$

Therefore,

$$\frac{1}{\cosh^2(2\vartheta d)} \leq 1, \quad (55)$$

and the factor B satisfies

$$B \geq 1 - 2\kappa\vartheta^2. \quad (56)$$

The first occurrence of degeneracy is therefore at

$$2\kappa\vartheta^2 = 1, \quad (57)$$

which corresponds to $d = 0$. At this point,

$$B = 0, \quad (58)$$

and the coefficient of the highest-order derivative vanishes.

8.3 Interpretation of the singularity

When the factor B vanishes, the differential equation loses its highest-order term. As a result, the system ceases to be of fourth order at that point, and the standard theory of elliptic equations no longer applies (Evans, 2010). This leads to the appearance of singular behaviour in the curvature equations, observed in the form of narrow necks in the membrane shape. From this perspective, the singularity is not merely a geometric feature of the surface, but reflects a degeneration of the differential operator governing the equilibrium.

8.4 Relation to curvature stiffness

The factor B can be interpreted as an effective stiffness coefficient associated with the deviatoric curvature mode. As the parameter $\kappa\vartheta^2$ increases, this coefficient decreases. At the threshold

$$2\kappa\vartheta^2 = 1, \tag{59}$$

the effective stiffness vanishes, and beyond this point it becomes negative in certain regimes. This behaviour mirrors the loss of positivity of the quadratic curvature kernel discussed in Section 4, and the loss of ellipticity of the linearised operator in Section 7. Representative numerical solutions of the same functional are shown in **Figure. 1**.

8.5 Conclusion (Euler–Lagrange viewpoint)

The degeneracy of the Euler–Lagrange equations occurs at the same threshold identified in the quadratic kernel and linearised operator analyses. This shows that the singularity observed in the axisymmetric formulation corresponds to the loss of ellipticity of the underlying geometric operator.

9 Unified interpretation and conclusion

We have analysed the curvature functional from three complementary viewpoints:

- the quadratic curvature kernel (energy level),
- the linearised operator (principal symbol),
- the Euler–Lagrange equations (nonlinear level).

Each of these approaches captures the behaviour of the system at a different level: energetic, differential, and fully nonlinear. Despite this difference, all three lead to the same structural condition. At the level of the quadratic kernel, the deviatoric eigenvalue vanishes and the energy loses coercivity. At the level of the linearised operator, the principal symbol becomes non-positive and ellipticity is lost. At the level of the Euler–Lagrange equations, the coefficient of the highest-order derivative vanishes, and the system becomes singular. These three phenomena therefore represent different manifestations of the same underlying transition. The neck singularity observed in the membrane shapes can thus be interpreted as a consequence of the loss of curvature stiffness in the deviatoric mode. This provides a consistent structural explanation that connects the variational formulation, the differential operator, and the nonlinear governing equations. The present framework clarifies the mathematical nature of the model and provides a basis for further extensions, including the study of stabilising mechanisms and nonlinear saturation effects.

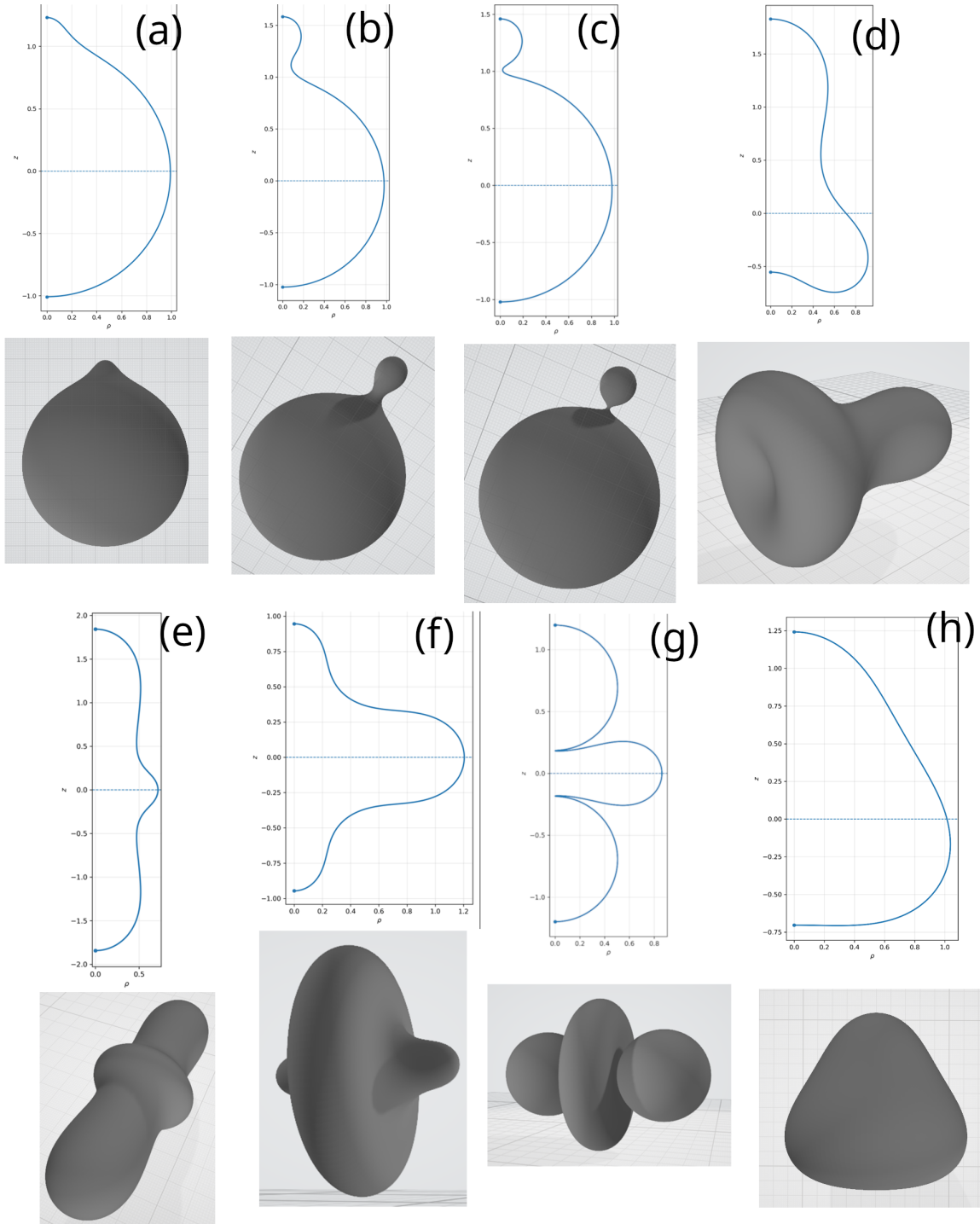


Figure 1: Representative axisymmetric equilibrium shapes obtained numerically from the analysed functional using a predictor-corrector scheme. Panels (a)–(h) show generating profiles $\rho(z)$ together with the corresponding reconstructed three-dimensional surfaces. The examples illustrate the diversity of stationary solutions obtained for different prescribed values of volume V and surface area A . The parameters of the entire shape given in panels (a) $v = 0.9920$, $\langle h \rangle = 1.0101$, (b) $v = 0.9500$, $\langle h \rangle = 1.1324$, (c) $v = 0.9500$, $\langle h \rangle = 1.1664$, (d) $v = 0.7325$, $\langle h \rangle = 1.1473$, (e) $v = 0.6950$, $\langle h \rangle = 1.2380$, (f) $v = 0.6950$, $\langle h \rangle = 1.1120$, (g) $v = 0.4950$, $\langle h \rangle = 1.7379$, (h) $v = 0.9500$, $\langle h \rangle = 1.0165$. Shape (b) data coordinates available at (Židánek et al., 2025). All shapes data on plots and 3D objects data available at (Židánek et al., 2026).

The singularity identified in the present analysis should not be interpreted merely as a numerical artefact or a consequence of parametrisation. Rather, it reflects a change in the structure of the governing equations. At the critical threshold, the coefficient of the highest-order curvature term vanishes, and the associated operator is no longer strictly elliptic. In this regime, the standard elliptic framework no longer applies, and the behaviour of solutions is governed by a degenerate structure. This suggests that the observed singular behaviour is linked to a loss of curvature stiffness in the deviatoric mode, although a detailed analysis of the resulting solution properties lies beyond the scope of the present work.

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